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COMMON FIXED POINT THEOREM IN COMPLEX VALUED-B-METRIC SPACE UNDER RATIONAL CONTRACTION

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ABSTRACT

In this paper, we proved a common fixed point theorem on complex valued b-metric space under rational contraction. The obtained result is an extension of some well known results in literature.

Keywords: *FIxed point-b-metric spaces, complex valued b-metric space.*
2000 Mathematics Subject Classification: 54H25, 47H10.

I. INTRODUCTION

In 1922, Banach [2]proved contraction principle which has wide application in many branches of mathematics such as mathematical analysis, computer sciences and engineering. In 1998, Czerwik [4] introduced the concept of b-metric space. In 2011, Azam et al.[1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. After the establishment of complex valued metric spaces, many researchers have contributed with their work in this space. Rouzkard and Imdad [10] generalized Azam et al.[1]. Subsequently Sintunavarat et al. ([14],[15]) obtained common fixed point results by replacing the constant of contractive condition to control functions. Singh et al.([11],[12],[13]) proved fixed point theorems in complex valued metric spaces. In this paper, we proved a common fixed point theorem on complex valued b-metric space under rational contraction.

II. PRELIMINARIES

Let C be the set of complex numbers and let $z_1, z_2 \in C$. Define a partial order $\leq$ on C
as $z_1 \leq z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.
It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied :
(1) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$
(2) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$
(3) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$
(4) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$

In particular, we will write $z_1 \leq z_2$ if one of (1), (2) and (3) is satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

Definition .2.1. Let X be a non empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to C$ satisfies the following conditions

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called complex valued-b-metric space or dq-b-metric space.

Definition.2.2. Let (X, d) be a complex valued b-metric space.

(1) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever $\exists\ 0 \le r \in C$ such that $B(x, r) = \{y \in X : d(x, y) \le r\} \subseteq A$.
(2) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.
(3) A subset $A \subseteq X$ is called closed whenever each element of A is a point of A.

Definition. 2.3. Let (X, d)be a complex valued b-metric space. Then a sequence $\{x_n\}^{\infty}_{n=1}$ in X is called a Cauchy's sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \ge n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition.2.4. Let (X, d)be a complex valued b-metric space. Then a sequence $\{x_n\}^{\infty}_{n=1}$ in X is called convergent sequence if and only if there exists $x \in X$ such that for all $n \in N$ for all $n > n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$, then we write $\lim_{n\to\infty} x_n = x$.

Definition 2.5. The complex valued b-metric space is complete if every Cauchy sequence convergent.

III. MAIN RESULT

Theorem 3.1

Let (X,d) be complete complex valued -b-metric space. Let $S,T : X \to X$ be a self mapping such that

$$\mathbf{d(Sx, Ty)} \le \frac{k[d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)]}{d(x,Ty) + d(y,Sx)} \quad \textbf{......(1)}$$

$\forall x, y \in X$, where $0 \le k < 1$ and $s \ge 1$. Then S and T have unique common fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that

$x_{n+1}=Sx_n$ and $x_{n+2}=Tx_{n+1}$ (2)

Consider

for n= 0, 1,2,3…

$d(x_{n+1}, x_{n+2}) = d(Sx_n, Tx_{n+1})$

$$d(x_{n+1}, x_{n+2}) \le \frac{k[d(x_n,Sx_n)d(x_n,Tx_{n+1})+d(x_{n+1},Tx_{n+1})d(x_{n+1},Sx_n)]}{d(x_n,Tx_{n+1})+d(x_{n+1},Sx_n)}$$

$$\le \frac{k[d(x_n,x_{n+1})d(x_n,x_{n+2})+d(x_{n+1},x_{n+2})d(x_{n+1},x_{n+1})]}{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}$$

$$\le \frac{k[d(x_n,x_{n+1})d(x_n,x_{n+2})]}{d(x_n,x_{n+2})}$$

$$\le kd(x_n, x_{n+1})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Continue this process we get,

$$d(x_{n+1}, x_{n+2}) \le kd(x_n, x_{n+1}), \ldots. \le k^n d(x_n, x_{n+1})$$

Now we show that $\{x_n\}$ is Cauchy sequence in X. Let m ,n in X and m > n.

Then we have

$$\begin{aligned} d(x_n, x_m) &\le s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)) \\ &\le sd(x_n, x_{n+1}) + s[s\{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\}] \\ &\le sk^n d(x_0, x_1) + s^2k^{n+1}d(x_0, x_1) + s^3k^{n+2}d(x_0, x_1) + \cdots \\ &\le sk^n d(x_0, x_1)[1 + sk + s^2k^2 + \cdots] \\ &\le \frac{sk^n}{1-sk} d(x_0, x_1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4) \end{aligned}$$

Then $\lim_{n\to\infty} d(x_n, x_m) = 0$, as limit $n, m \to \infty$, since $k < 1$, $\lim_{n\to\infty} \frac{sk^n}{1-sk} d(x_n, x_m) = 0$

Hence $\{x_n\}$ is Cauchy' sequence in X. Since X is complete, so it converges to u.
Now we show that u is fixed point of S. If not then there exist z in C such that
d(Su, u) = z > 0 from (1) we get ,

$$d(u,Su) = z$$
$$\leq s[d(u,x_{n+2}) + d(x_{n+2},Su)]$$
$$\leq s[d(u,x_{n+2}) + d(Su,Tx_{n+1})]$$
$$\leq sd(u,x_{n+2}) + s[\frac{k[d(u,Su)d(u,Tx_{n+1})+d(x_{n+1},Tx_{n+1})d(x_{n+1},Su)]}{d(u,Tx_{n+1})+d(x_{n+1},Su)}]$$
$$\leq sd(u,x_{n+2}) + s[\frac{k[d(u,Su)d(u,x_{n+2})+d(x_{n+1},x_{n+2})d(x_{n+1},Su)]}{d(u,x_{n+2})+d(x_{n+1},Su)}]$$

Now taking as limit n → ∞ , we get z < 0 , a contradiction .Therefore Su = u .
This implies that u is a fixed point of S.
Now we show that u is fixed point of T. If not then there exist z in C such that
d(u, Tu) = z > 0 from (1) we get,

$$d(u,Tu) = z$$
$$\leq s[d(u,x_{n+1}) + d(x_{n+1},Tu)]$$
$$\leq s[d(u,x_{n+1}) + d(Sx_n,Tu)]$$
$$\leq sd(u,x_{n+1}) + s[\frac{k[d(x_n,Sx_n)d(x_n,Tu)+d(u,Tu)d(u,Sx_n)]}{d(x_n,Tu)+d(u,Sx_n)}]$$
$$\leq sd(u,x_{n+2}) + s[\frac{k[d(x_n,x_{n+1})d(x_n,Tu)+d(u,Tu)d(u,x_{n+1})]}{d(x_n,Tu)+d(u,x_{n+1})}]$$

Now taking as limit n → ∞ ,we get z < 0 , a contradiction .Therefore Tu = u .
This implies that u is a fixed point of T.

Uniqueness
Now we show that S and T have unique common fixed point. Consider u and v are two fixed point of S and T. Since Su = u, Sv = v, Tu
= u and Tv = v. then

$$d(u,v) = d(Su,Tv)$$
$$\leq \frac{k[d(u,Su)d(u,Tv) + d(v,Tv)d(v,Su)]}{d(u,Tv) + d(v,Su)}$$
$$\leq \frac{k[d(u,u)d(u,v) + d(v,v)d(v,u)]}{d(u,v) + d(v,u)}$$
$$\leq 0.$$

Hence u = v .This implies that S and T have unique fixed point.

Corollary 3.2.
Let (X, d) be complete complex valued -b-metric space. Let T : X → X be a self mapping such that

$$d(Tx,Ty) \leq \frac{k[d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)]}{d(x,Ty) + d(y,Tx)}$$

∀x, y ∈ X, where 0 ≤ k < 1 and s ≥ 1. Then T has unique fixed point.
Proof: Put S = T in the above theorem 3.1,we get the result.

Corollary 3.3.
Let (X,d) be complete complex valued -b-metric space. Let T : X → X be a self mapping such that

$$d(T^n x,T^m x) \leq \frac{k[d(x,T^n x)d(x,T^n y) + d(y,T^n y)d(y,T^n x)]}{d(x,T^n y) + d(y,T^n x)}$$

∀x, y ∈ X, where 0 ≤ k < 1 and s ≥ 1. Then T has unique fixed point.

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